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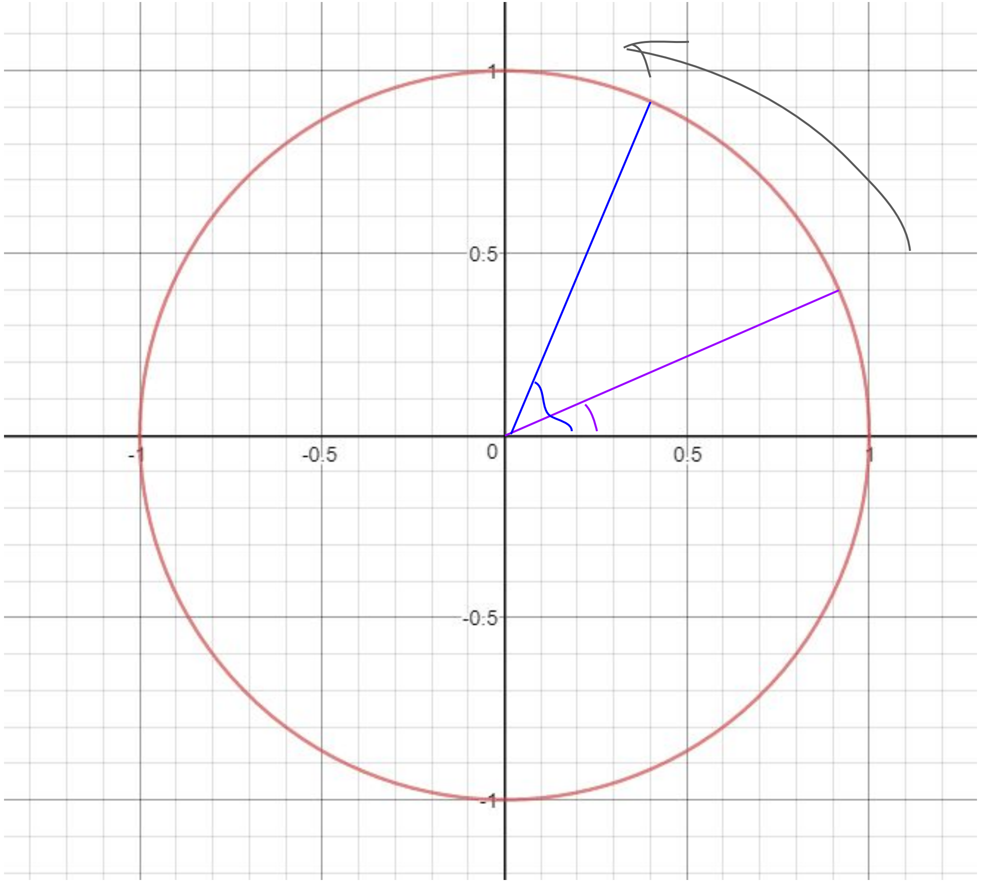
Understanding the Trigonometric Circle

The “Unit Circle”



The special thing about the trigonometric circle is its radius=1.
You will see in this lecture that the **radius of the circle** and the **hypotenuse** are one and the same as long as you use the conventions enforced in the trigonometric circle.

The **radius value of 1** simplifies things by removing the hypotenuse in all calculations as follows. Let's say the hypotenuse = h, then:

a/h = a
b*h = b,
so everything is simpler due to **r = h = 1**.

CONVENTIONS OF THE Trigonometric Circle

Angles are drawn in the circle from the (0,0) point.

Angles grow in counter-clockwise direction.
In other words these are angles in standard position.

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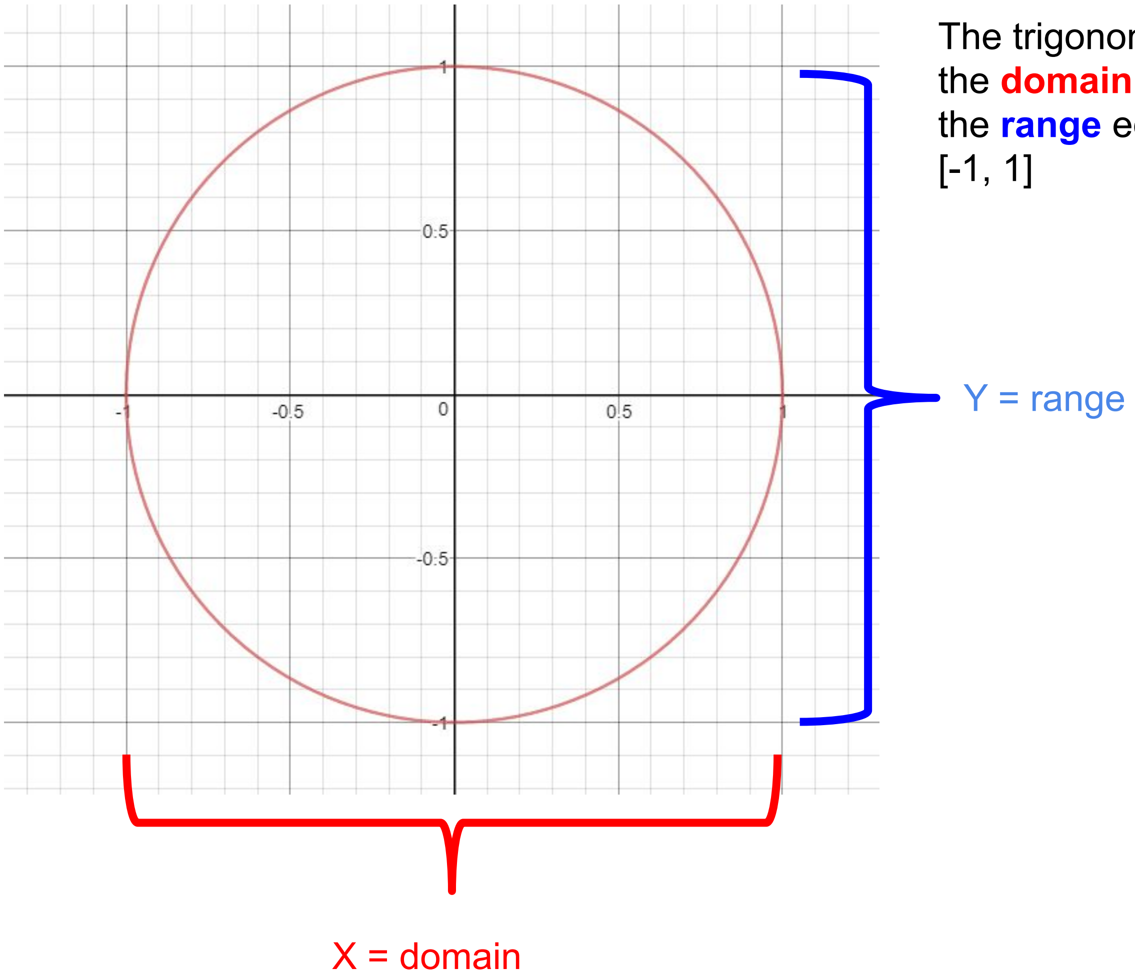

The trigonometric circle has both the **domain**, and the **range** equal to [-1, 1]

Reminder:
Domain = all allowed x values,
Range = all allowed y values.

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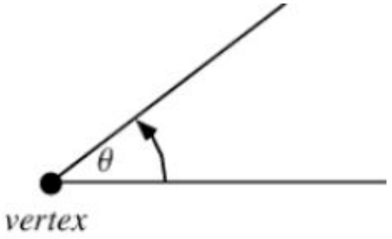
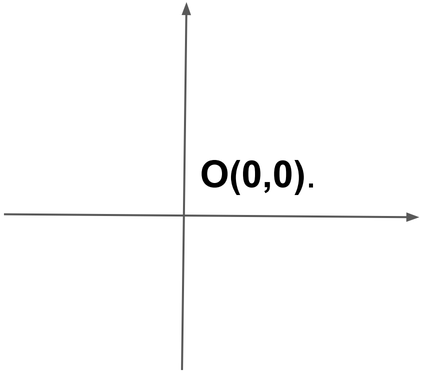
Just like we designate the letter **x** for unknowns we solve for in equations, in trigonometry we represent angles by various Greek letters, most commonly used are:

ϴ “Theta”
α “Alpha”
β “Beta”

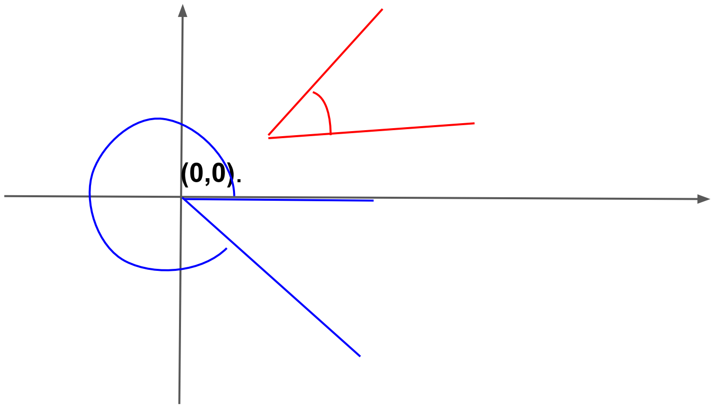
Terms

The point about which an **angle** is measured is called the **angle's vertex**.





The point where two axes meet is called the **Origin** and is usually denoted as **O(0,0)**.



The blue angle is in “standard position”.
The red angle is not.

Placing the vertex of an angle in the axes’ origin is called “*placing it the angle in the standard position*”.
The standard position relates to the origin but says nothing about the rotation of the angle.

Angle Rotation

Normal counter-clockwise rotation.
This means the angle is a **positive** angle.

Clockwise rotation, opposite to the usual way that angles grow in the trigonometric circle.
This means the angle is **negative**.
The following relation can fix the angle back into the positive range:

"Positive version of $\boldsymbol{\beta}$" = 360° - $|\boldsymbol{\beta}|$

Reminder: **|x| = the absolute value of x**, that is the value of the number x but **without** its negative sign if present, in other words |x| is ensured to be the positive version of the number x.

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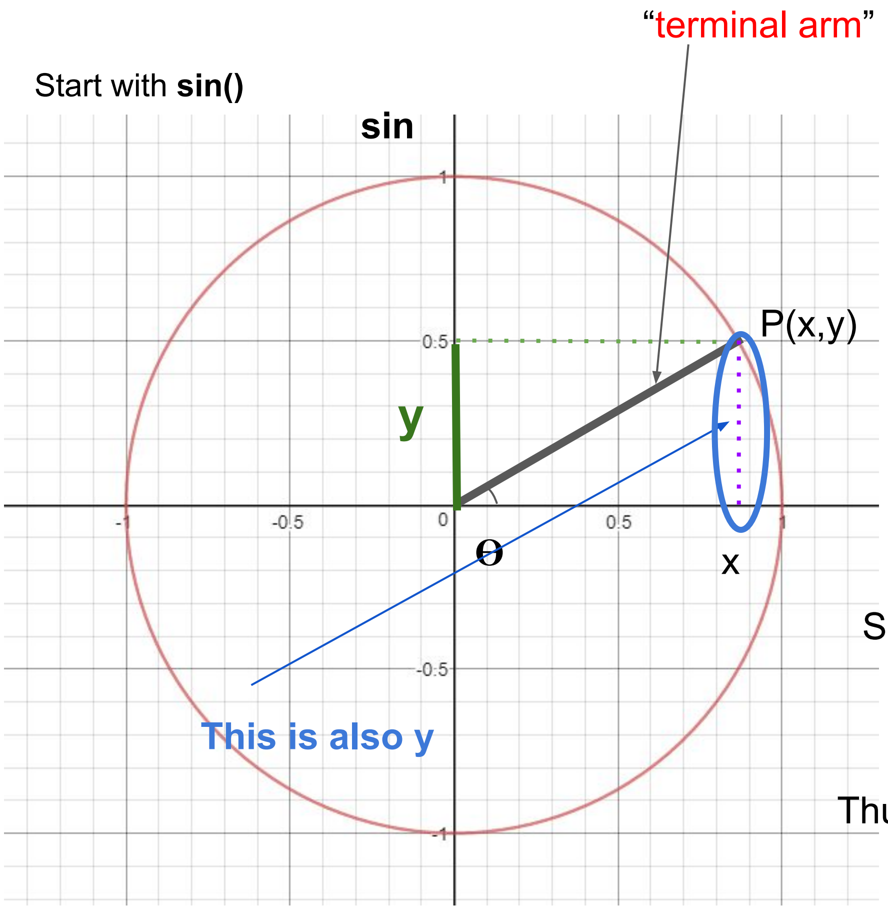
Trigonometric Functions in the Unit Circle

Start with **sin()**



Draw angles from the center, by extending a radius.
Stop on the rim of the circle.

This line is called the "terminal arm" and **is equivalent to a radius**.

Mark the point on the rim as P. It must have an (x, y).
Angles can grow by sliding P on the rim counter-clockwise.

Project P on the two axes, i.e. drop perpendiculars to both axes.

Sin θ is the "soh" in **soh**cahtoa.

Sin θ = opposite cathetus over the hypotenuse

Thus Sin θ = y/1

Sin θ = y

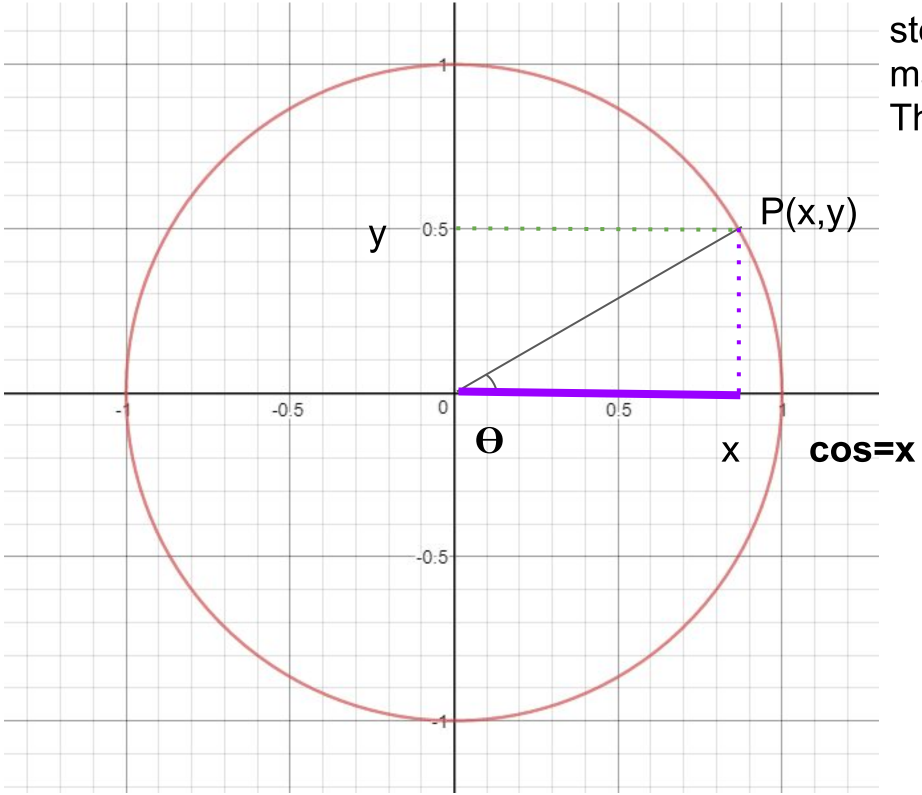
y

It's the radius and is = 1.

In this circle: sin and y are the same. Mark the y -axis with a **sin**.

Trigonometric Functions in the Circle

The **cos()** function



Like before, draw the terminal arm from the center, stop on the rim,
mark the point on the rim as P. It has its own (x, y).
Then project P on the two axes.

Cos **ϴ** is "cah" is soh**cah**toa

cos **ϴ** = adjacent cathetus over the hypotenuse

x

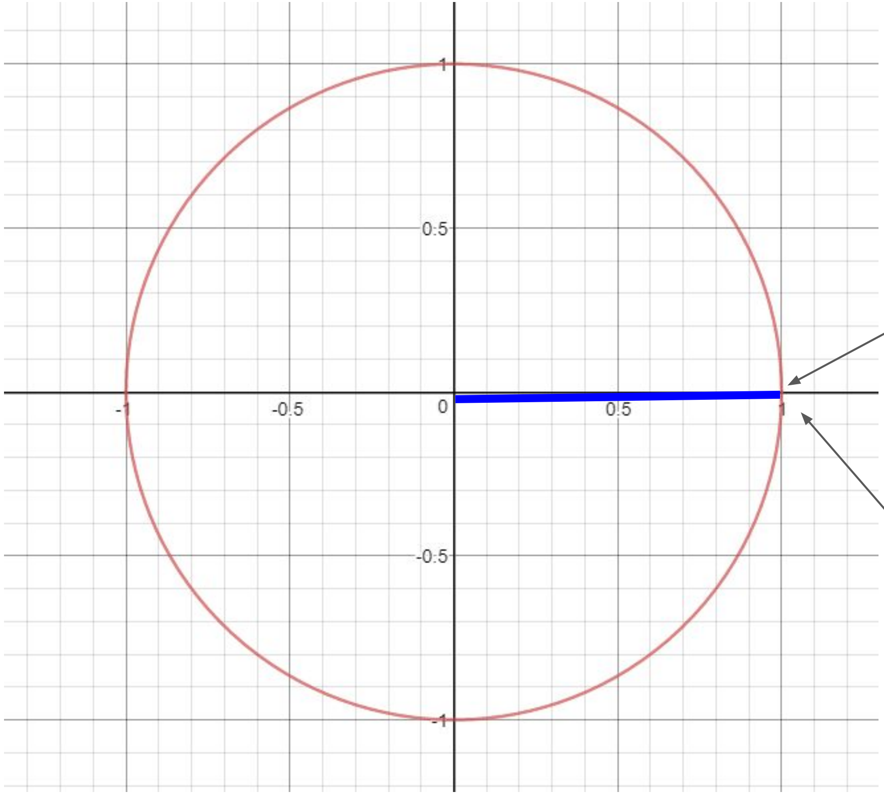
It's the radius and is = 1.

Thus cos **ϴ** = x/1

Cos **ϴ** = x

In this circle: cos and x are the same. Mark the x -axis with a **cos**.

Where is P(x,y) for the angle θ = 0°?



P(x,y) =P(x,0)

This is x, what is y at point P?

P's y here is = 0. This angle "has no height".

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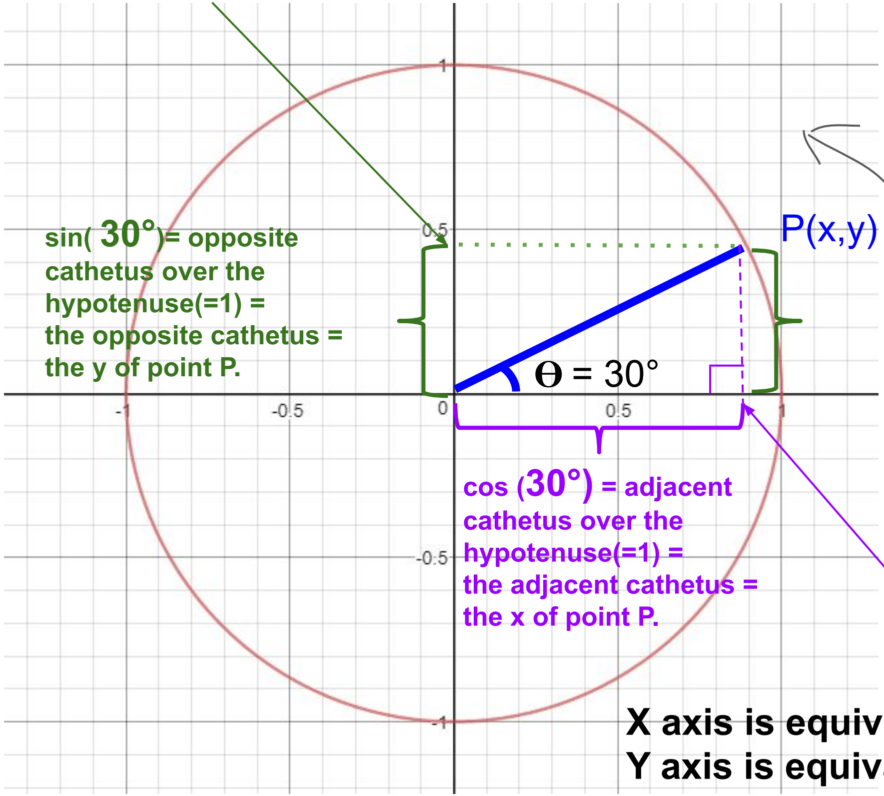
The trigonometry of is 30° (using a calculator):
sin (30°) = 0.5
cos(30°) = 0.866
and in the unit circle...

Where is P(x,y) for the angle Θ = 30°?

This is also sin(30°) = 0.5
b/c Θ lives in the same right angled triangle.



What are the coordinates (x,y) of point P?

= (0.866, y)

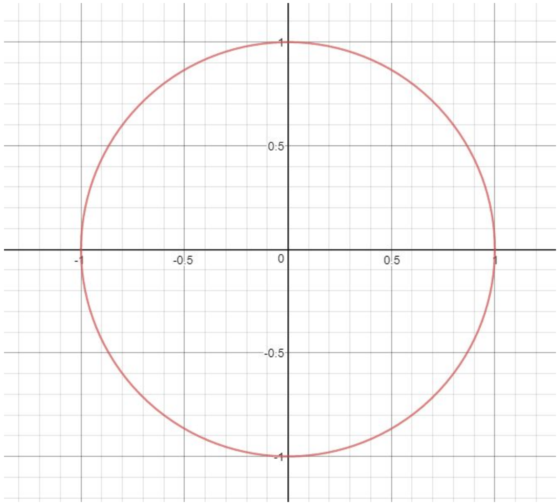
= (0.866, 0.5)

To get a larger angle **slide** P in a counterclockwise position but keep it on the rim.

This is also cos(30°) = 0.866
b/c Θ lives in that right angled triangle.

X axis is equivalent to the cos() values of the angles in the circle.
Y axis is equivalent to the sin() values of the angles in the circle.

Can you mark the following angles on the circle?



0°

90°

180°

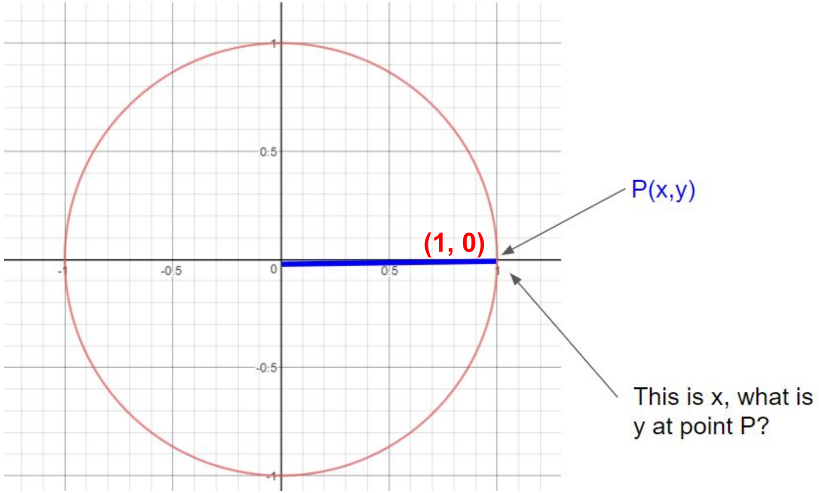
270°

360°

Example

P of the angle θ = 0°

What are the coordinates of these angles for their P(x,y) on the rim of the circle?



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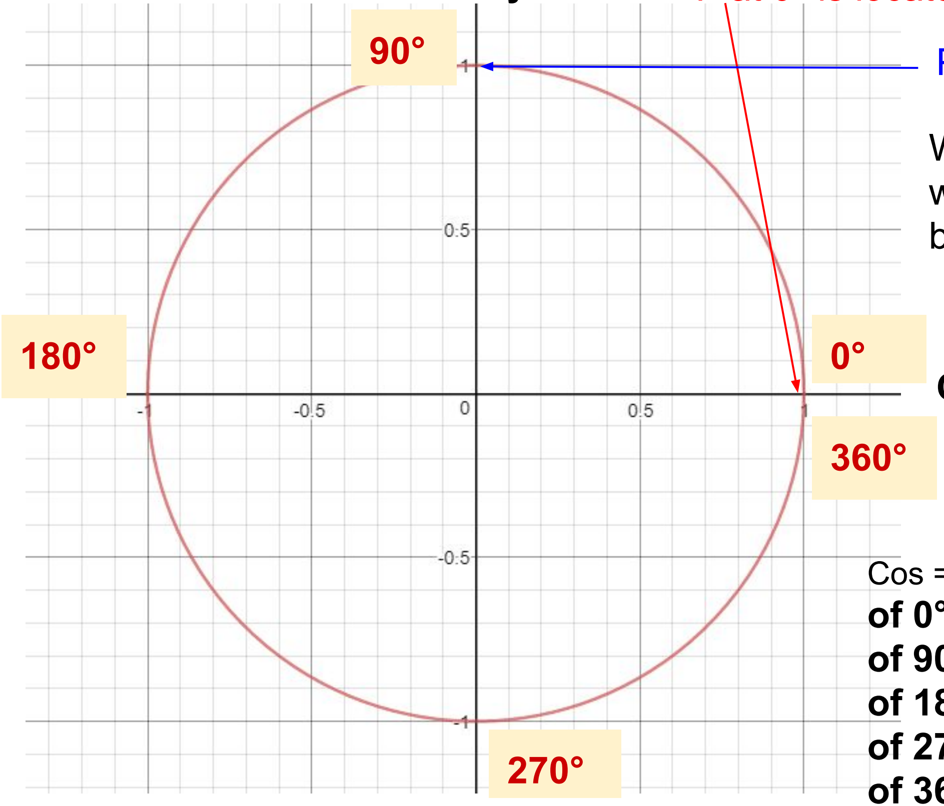
Mark a few angles on the circle: 0, 90, 180, etc.

And imagine we have renamed
X axis to cos(), and
Y axis to sin(), but the values have not changed, only the name of the axes.

Sin = y

P at 0° is located at the left where (x,y) = (1, 0)

P at 90° is located at the top where (x,y) = (0, 1)



We won’t say (x, y) anymore,
we are renaming (x,y) to (cosΘ, sin Θ)
but the values stay the same as (x,y). I.e.:
at 0° is (cos 0°, sin 0°) = (1, 0) and,
at 90° is (cos 90°, sin 90°) = (0, 1).

Cos = x

Maintaining the positions of the previously x and y.

Read the **x** of the angles as cos, and the **y** as sin. Remember to keep P right on the rim, never off it.

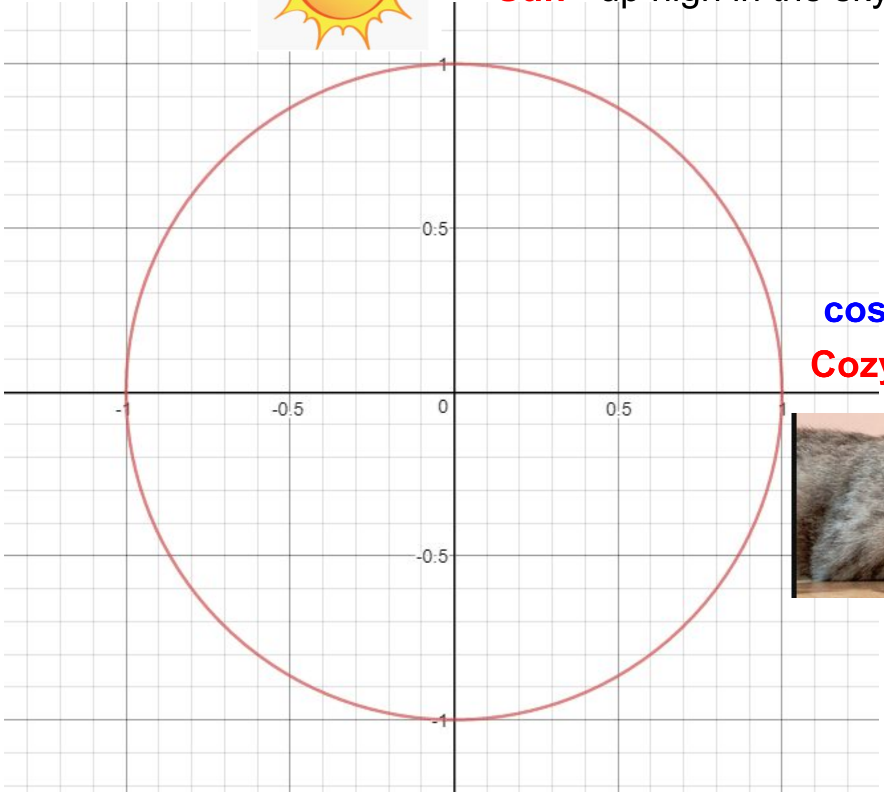
Cos = the same as x
of 0° = 1
of 90° = 0
of 180° = -1
of 270° = 0
of 360° = 1

sin = the same as y:
of 0° = 0
of 90° = 1
of 180° = 0
of 270° = -1
of 360° = 0

Remembering the axes of the Unit Circle

Sin

Sun - up high in the sky



cos

Cozy - flat on the floor

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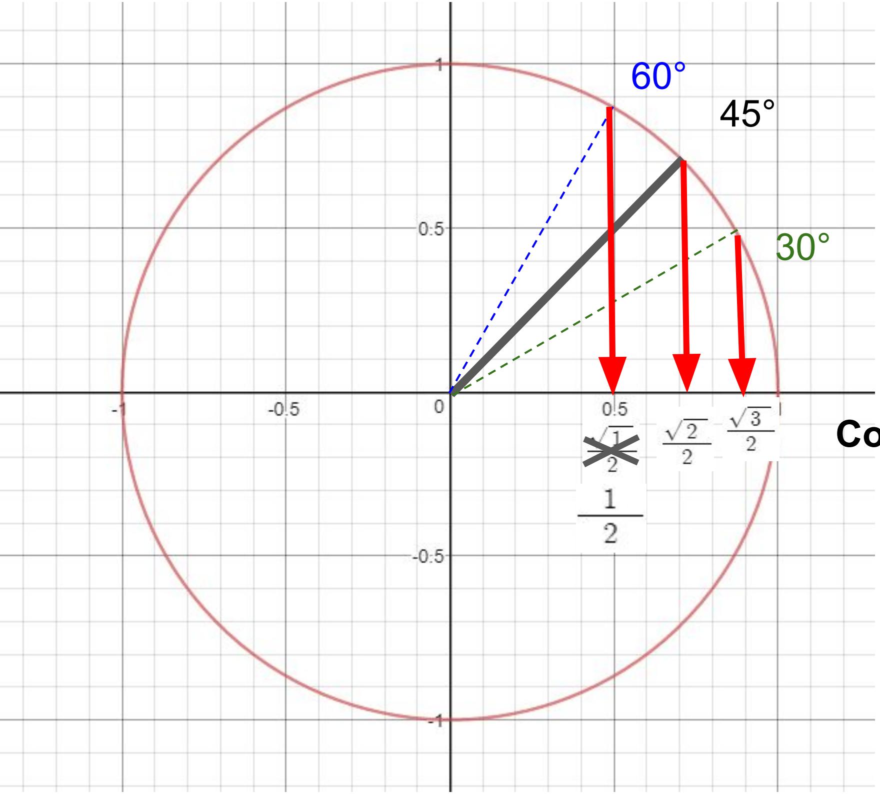
Let's reason, and
partially memorize
values for sin, and cos
for other not so obvious angles: **30°, 45°, 60°**



You saw before the values of **cos** are the projections onto the x axis.

You can already see whose x-s are larger and smaller: *values grow from left to right.*

Interestingly, these three angles (**30°, 45°, 60°**) always work with the following **square roots** halved, in both the cos(), and the sin() functions:

$$\frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}$$

If you knew the three cos x-values are the above,
by what logic would you assign them to the red arrows?

Cos = x

We know that: $\sqrt{1} < \sqrt{2} < \sqrt{3}$

Then their halves must be sorted so too

$$\frac{\sqrt{1}}{2} < \frac{\sqrt{2}}{2} < \frac{\sqrt{3}}{2}$$

There is only one way to place the values so to makes sense:
arrange them on the red arrows in sorted order from left to right.

One more thing: replace $\sqrt{1}$ with 1, since they are equal.

<u>CONCLUSION</u>

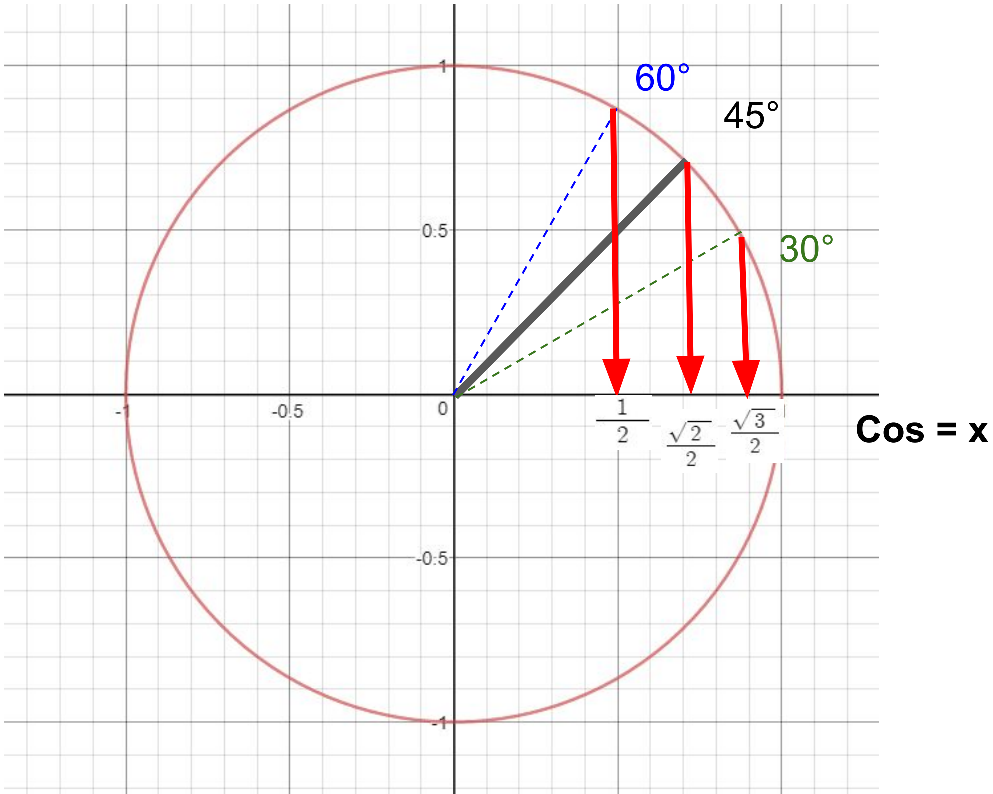
Memorizing cos() values is tricky and hard to remember.
You only need to remember it's always these 3 values for these 3 angles **30°, 45°, 60°**, and you only have to use your logic on how to sort them on their number lines.

$$\frac{1}{2} < \frac{\sqrt{2}}{2} < \frac{\sqrt{3}}{2}$$

Always draw the trig circle
sort the values in growing order.
Then read cos() values off the circle.

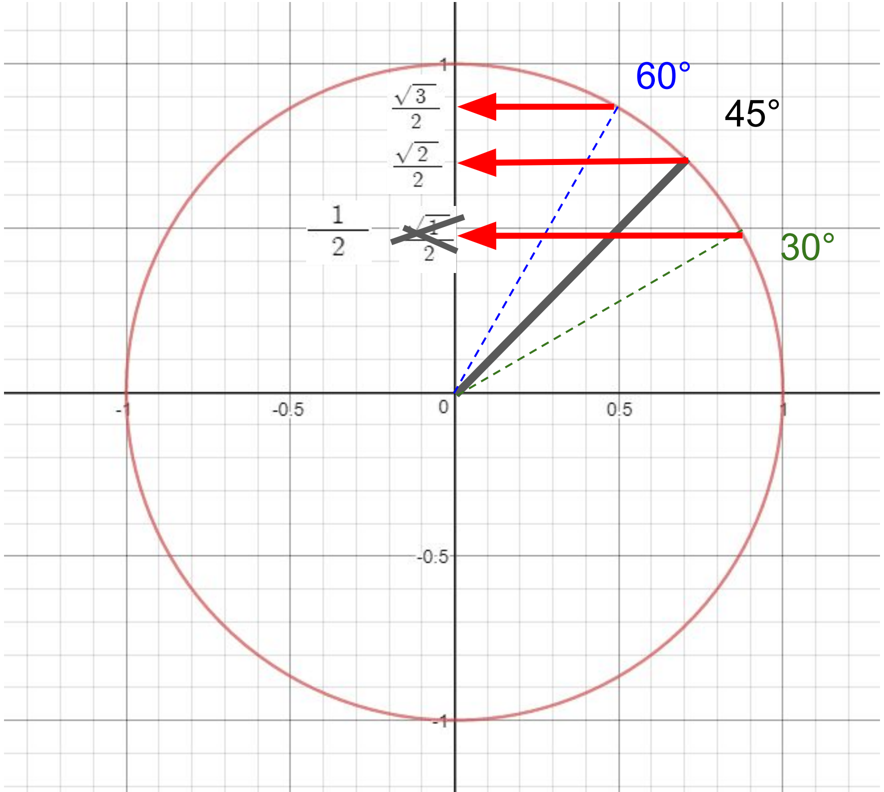
cos:

30°	$\frac{\sqrt{3}}{2}$
45°	$\frac{\sqrt{2}}{2}$
60°	$\frac{1}{2}$



Let's do the same for sin=y, again this is only for the angles: **30°, 45°, 60°**

sin = y



Same idea: the three angles always work with the following **square roots** halved:

$$\frac{\sqrt{1}}{2} < \frac{\sqrt{2}}{2} < \frac{\sqrt{3}}{2}$$

You can only sort them:

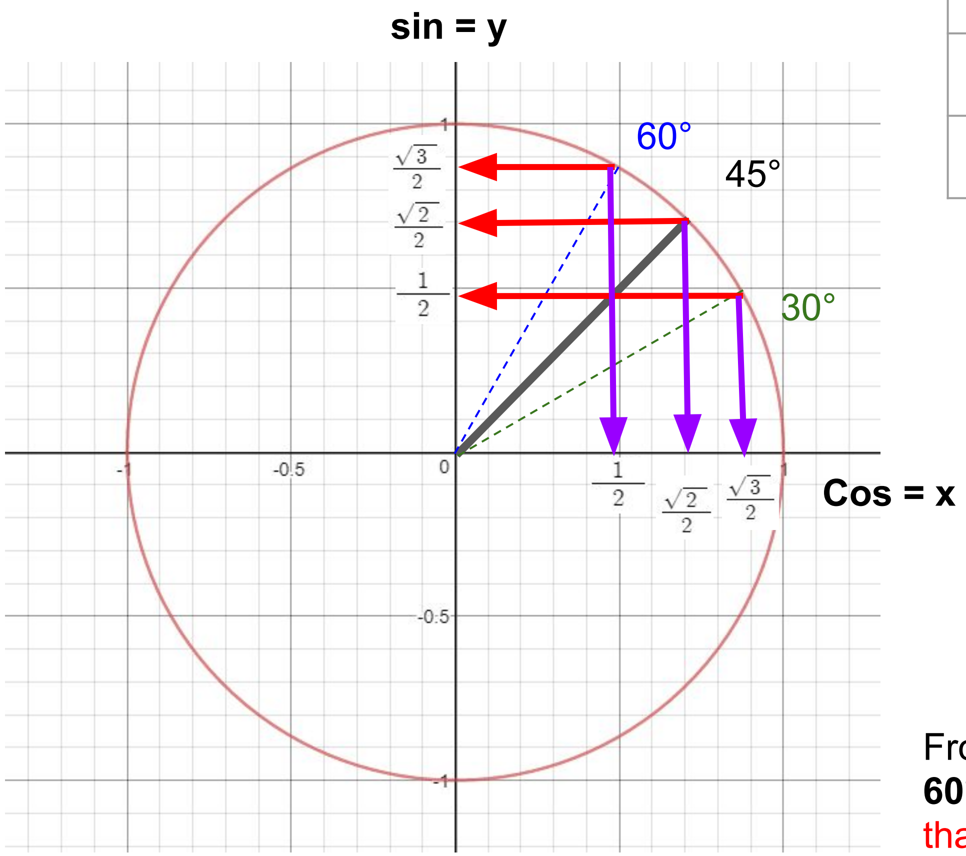
- largest at the highest
- smallest at the bottom

Then you have to conclude:

sin

30°	$\frac{1}{2}$
45°	$\frac{\sqrt{2}}{2}$
60°	$\frac{\sqrt{3}}{2}$

List them all together



	sin	cos
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$

It is important to recognize these values in the future.
Recognize them so you can work in reverse like so:
Sin $\boldsymbol{\theta}$ = $\frac{\sqrt{3}}{2}$

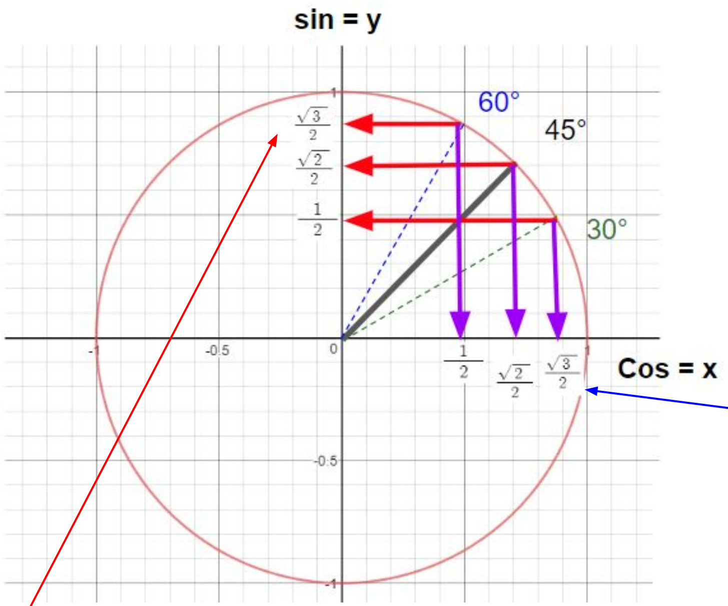
Notice $\boldsymbol{\theta}$ and $\frac{\sqrt{3}}{2}$ have swapped position in the **inverse** function.

Use your calculator:

$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \boldsymbol{\theta}$

From the result on your calculator conclude $\boldsymbol{\theta}$ is **60°** and other **matching** angles, see next what that means.

Here are some examples of what things you should be able to recognize by now



Some trigonometric function yields $\frac{\sqrt{3}}{2}$

If we are talking about cos then we are looking at a **30°** angle.

If we are talking about sin then we are looking at a **60°** angle.

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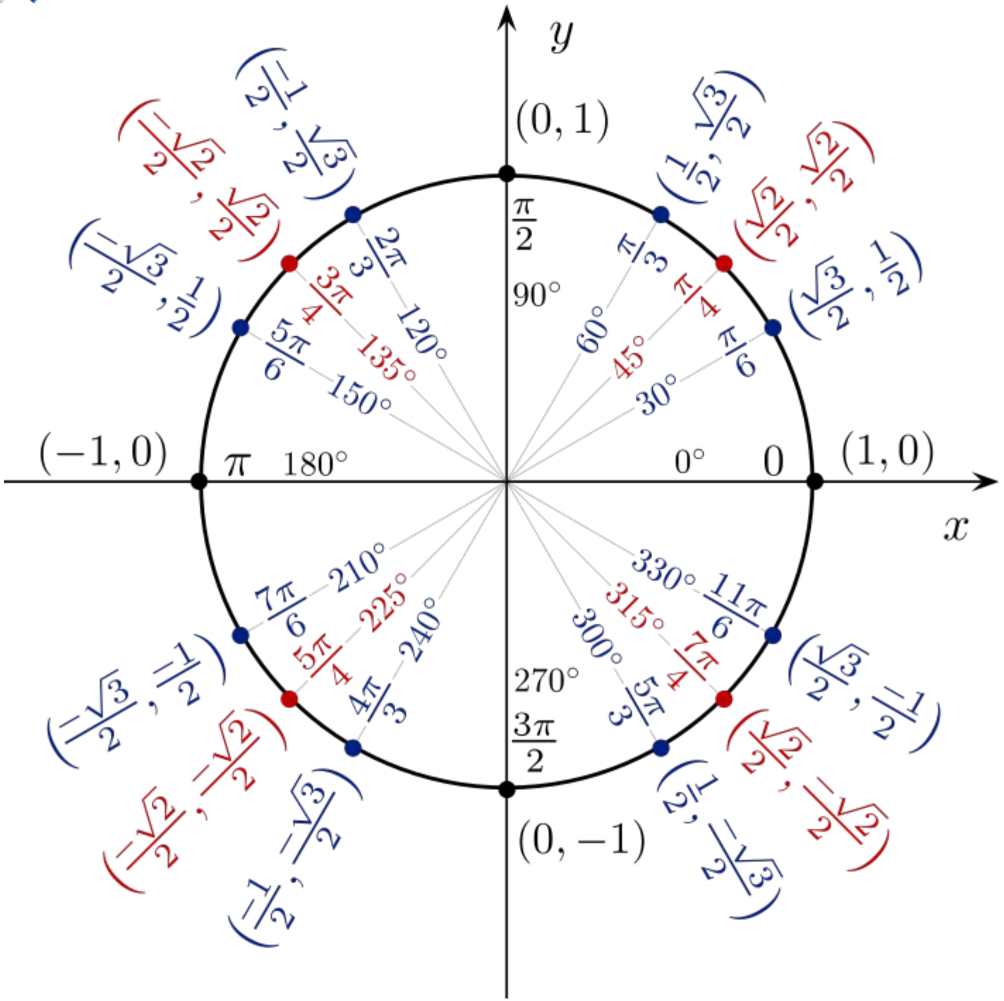


Source and credit
https://en.wikipedia.org/wiki/Unit_circle

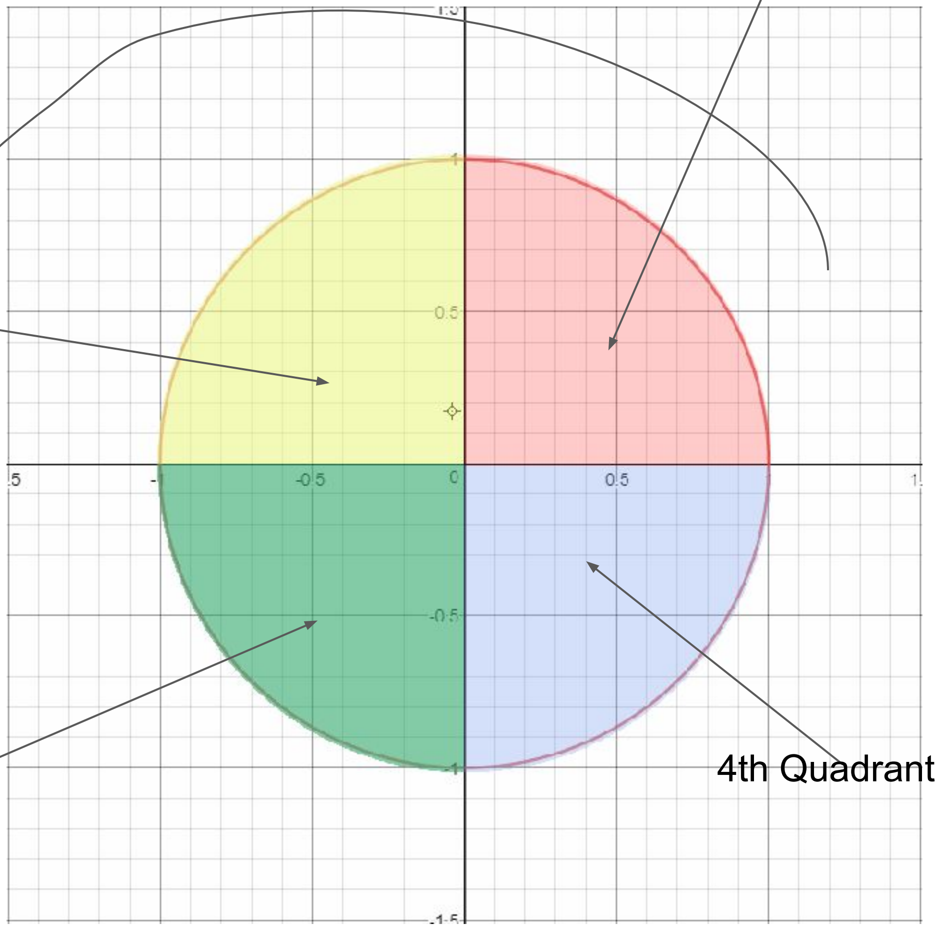
Also remember that angles reflect in the other quadrants and thus have to have matching values and at times with different signs. We’ll talk more about this in the next slides.

1st Quadrant

First quickly, what are quadrants?

The 2 axes divide the circle into 4 parts - each colored differently here

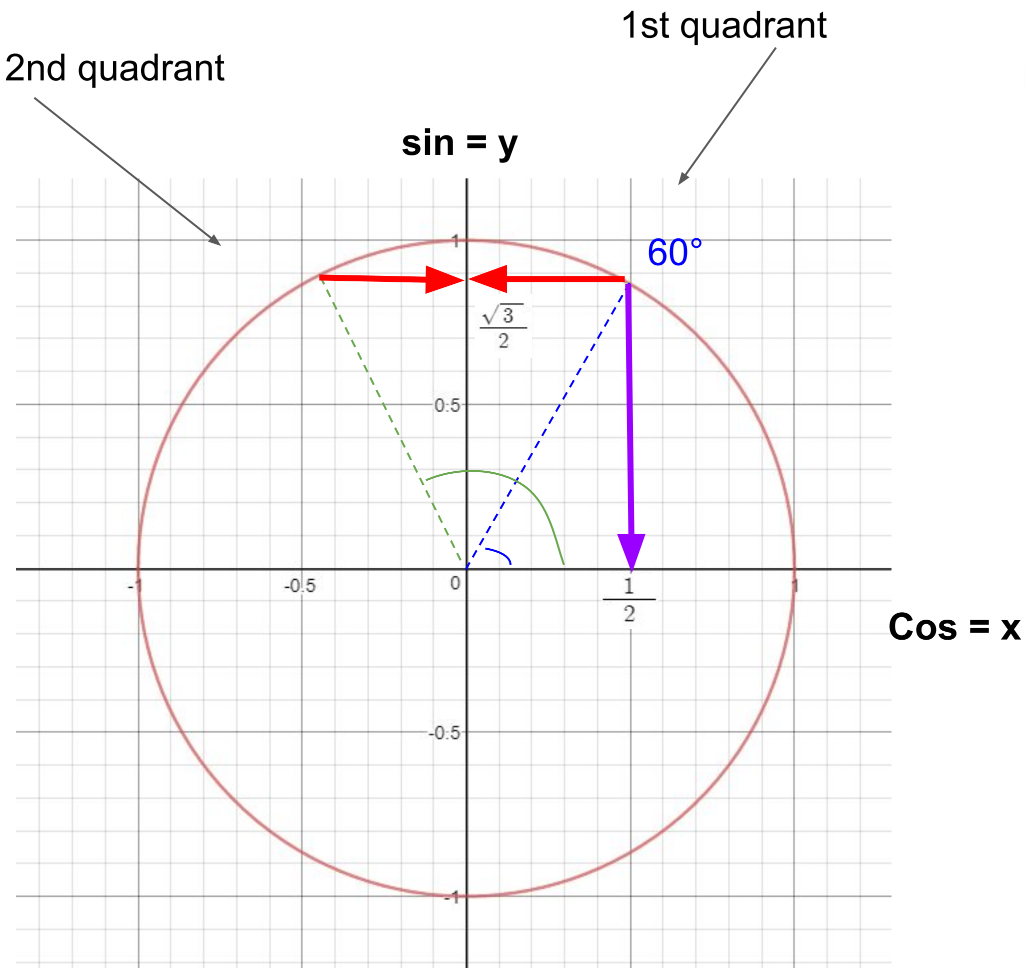
2nd Quadrant



Count counter-clockwise.

4th Quadrant

3rd Quadrant



For example 60°

Is there a matching angle
in the second quadrant?

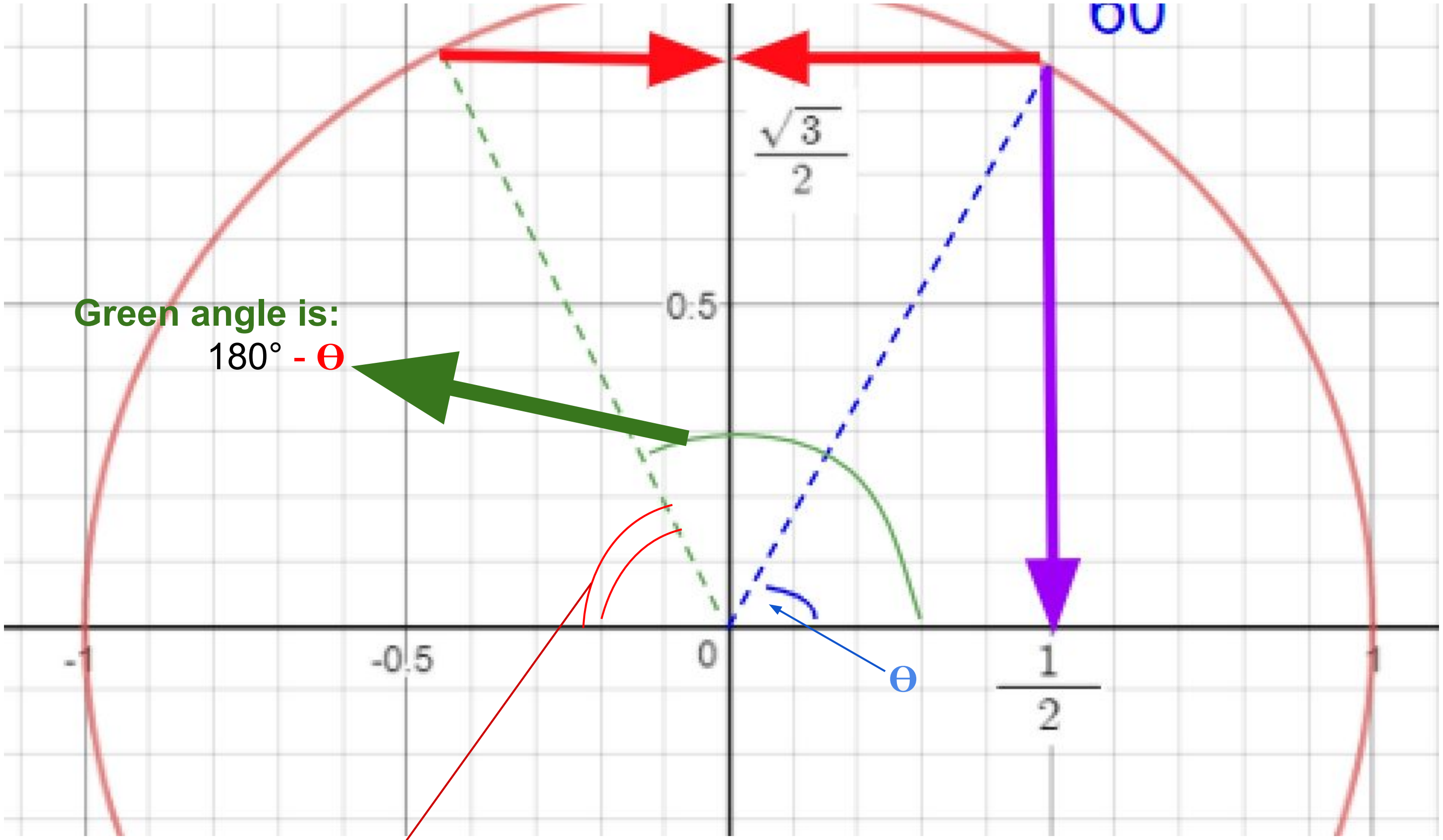
One that has the same y?

Reverse engineer such an angle.
Use the mirror image across y-axis.

The green and blue angles have the same y, and thus the same sin.

If blue is **Ɵ**, what is the green one?

Next page
zooms in...



Thus: an equivalent sin (or y) value for **θ**,
is the angle:
180° - **θ**
In our example **θ** was 60°.
So the blue **θ** and the green angle 180°-60° = 120°, both have a positive sin() value of:

$$\frac{\sqrt{3}}{2}$$

Also **θ** due to symmetry

What about the 4th and the 3rd quadrants?

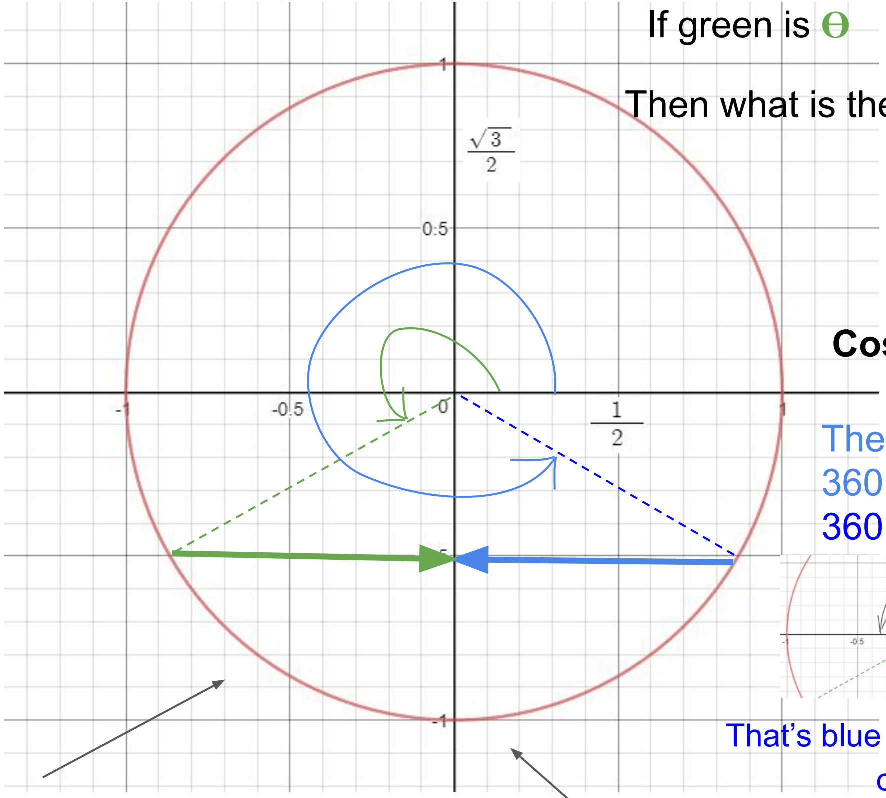
sin = y

Reverse engineer such an angle.
Use the mirror image across y-axis.

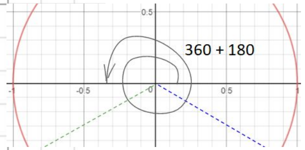
If green is θ

Then what is the blue?

Cos = x



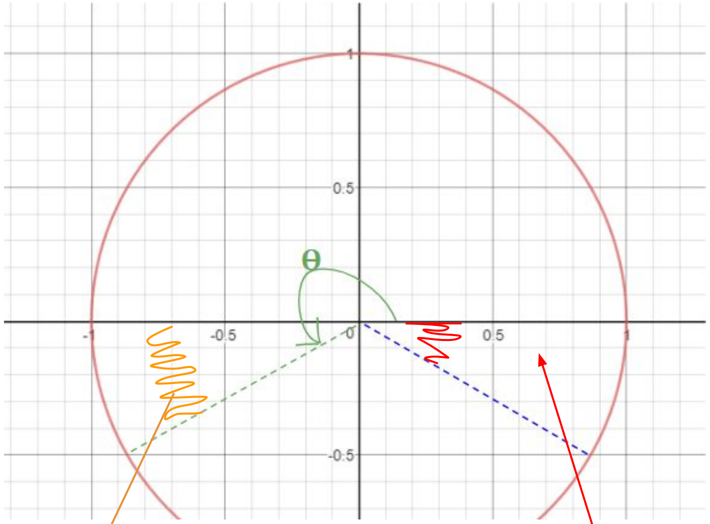
Then the blue is
$360 - (\theta - 180) =$
$360 + 180 - \theta$



That's blue = 540 - θ,
or 180 - θ

3rd quadrant

4th quadrant



Then this is θ - 180

Then this is θ - 180
too, due to symmetry

Thus for every angle in the 3rd Q (green) there has to be an angle in the 4th Q (blue) that has the same negative y (sin) values. Example next:

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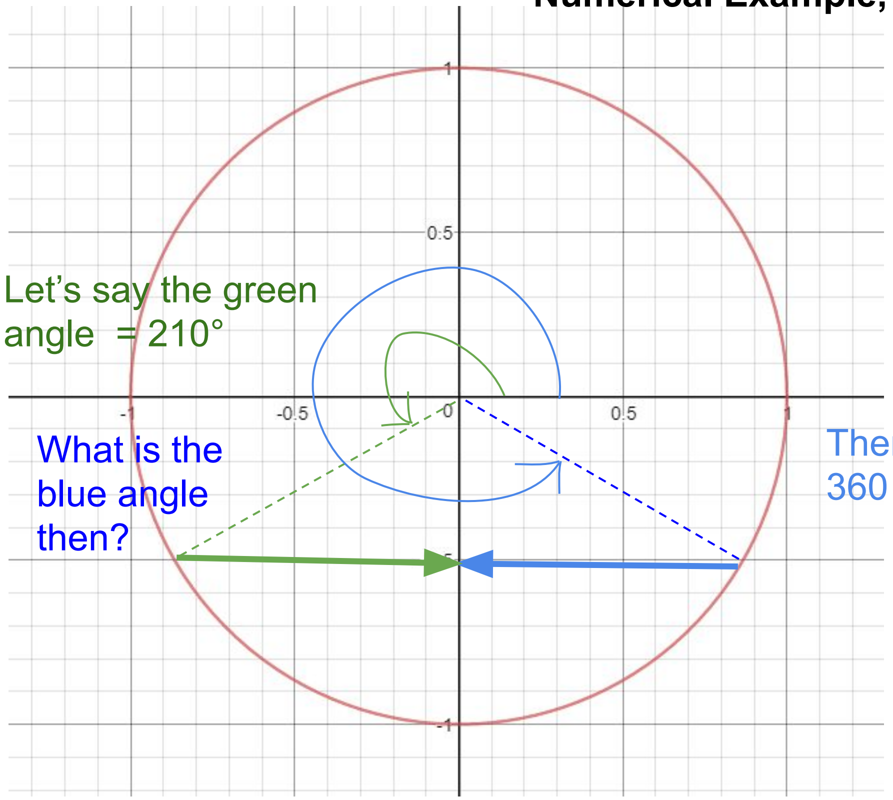
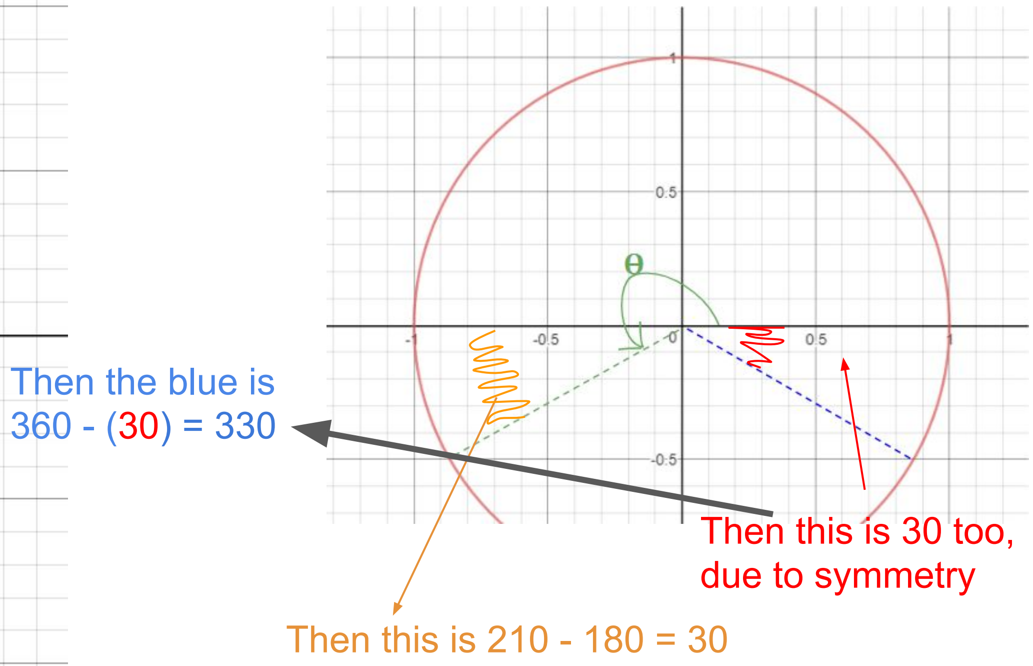
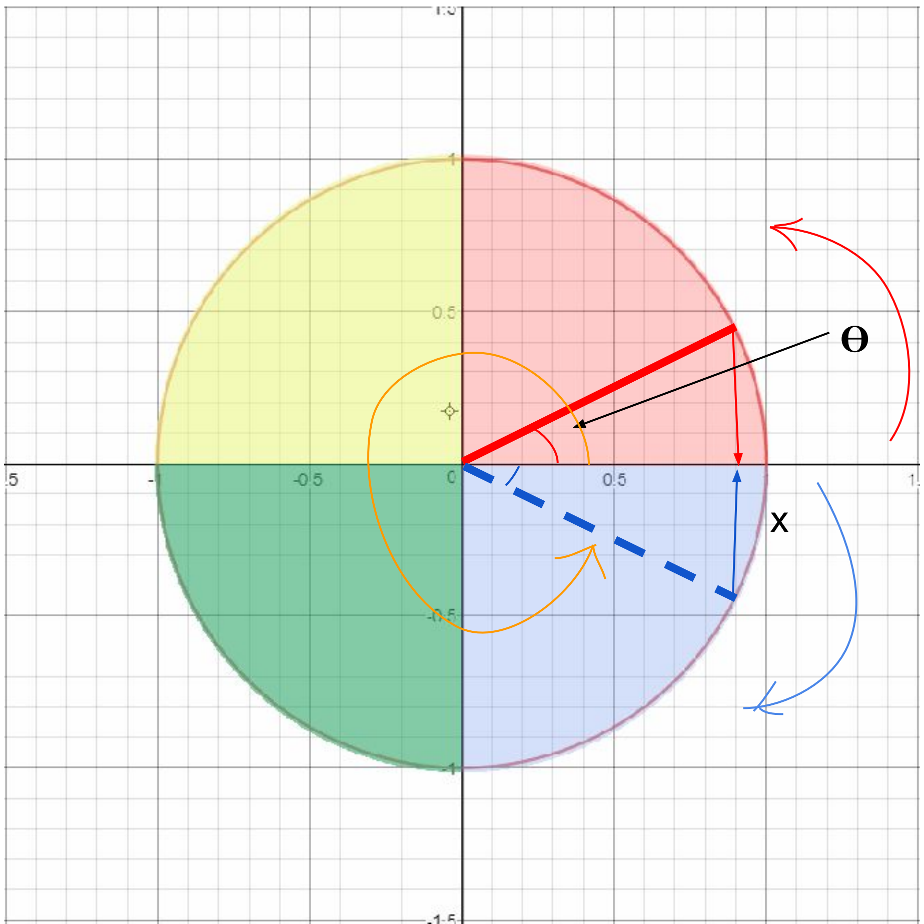
Numerical Example, Calculating Angles...





You can do the same with **cos**. Angles in the 1st quadrant have matching cos values in the 4th quadrant.



Both the red and blue values project their x (or cos) onto the same positive x position on the x axis. If the 1st quadrant angle is θ

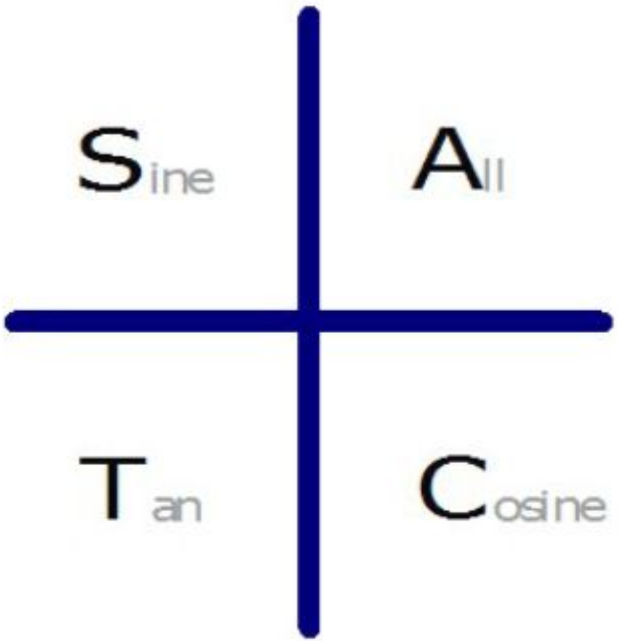
Then the 4th quadrant one is $-\theta$, and rotates in the opposite direction

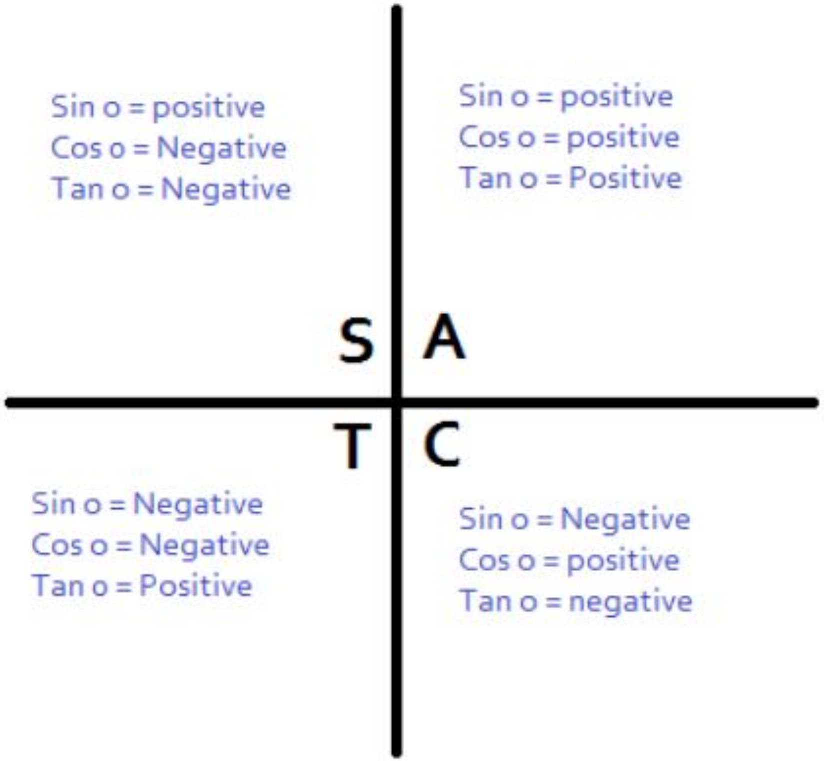
We normally do not refer to angles as negative, so instead fix the rotation to reflect a normal counter-clockwise direction

"$-\theta$" = $360° - \theta$

What is **CAST**?

This is where the values of the functions listed are is positive based on quadrants.





It is better to understand based on the previous slides. Jot a little trigonometric circle and reason your way from there.

Remembering CAST incorrectly will end up in the wrong answers… I don't like to use it!

Don’t forget...

Whenever solving an equation, especially if you are told the range is [0°, 360°], or stated as
$0° <= \theta <= 360°$

You must always look for matching angles in other quadrants to complete the solution. Without it only partial points will be given to your solution.

End of slide show.